BI-ELEMENT REPRESENTATIONS OF TERNARY GROUPS

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Dedicated to the memory of our coauthor and friend
Władek Marcinek who unexpectedly passed away on June 9, 2003

Abstract. General properties of ternary semigroups and groups are considered. The bi-element representation theory in which every representation matrix corresponds to a pair of elements is built, connection with the standard theory is considered and several concrete examples are constructed. For clarity the shortened versions of classical Gluskin-Hosszú and Post theorems are given for them.

1. Introduction

Ternary and n-ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. Firstly ternary algebraic operations were introduced already in the XIX-th century by A. Cayley. As the development of Cayley's ideas it were considered n-ary generalization of matrices and their determinants [22, 12] and general theory of n-ary algebras [15, 3] and ternary rings [16]. For some physical applications in Nambu mechanics, supersymmetry, Yang-Baxter equation, etc. see e.g. [14, 23, 18]. The statement here is something different and based on our previous preliminary report [2], where also ternary algebras and ternary Hopf algebras were considered.

The notion of an n-ary group was introduced in 1928 by W. Dörnte [5] (inspired by E. Nöther) which is a natural generalization of the notion of a group and a ternary group considered by Certaine [4] and Kasner [13].

In this paper we reformulate necessary material on ternary semigroups and groups [1, 20] in abstract language.

In the (binary) group theory representations were introduced as a matrix realization of group elements and abstract group action by usual matrix multiplication, when one element was described by one matrix. In this paper we propose a new bi-element approach to the representation theory of ternary group, when one matrix parametrizes two elements of a ternary group.

An alternative approach to ternary group representations was made in [24], where it was proposed, instead of our operator-valued functions of two variables, functions of one variable taking value in a pair of operators (matrices), viz. $\Pi^{wan} : G \to (\Pi_1^{wan}(x), \Pi_2^{wan}(x)) \in GL(V) \times GL(V)$ with another analog of homomorphism. Unfortunately, in [24] there were no given concrete examples nor connections with derived case.

Here, using our method, we present several concrete examples and consider connection with binary case proving the classical Gluskin-Hosszú and Post theorems.

2. Ternary semigroups

A non-empty set G with one *ternary* operation $[\;]: G\times G\times G\to G$ is called a *ternary groupoid* and is denoted by $(G,[\;])$ or $\left(G,m^{(3)}\right)$. We will present some results using second notation, because it allows to reverse arrows in the most clear way. In proofs we will mostly use the first notation due to convenience and for short.

If on G there is a binary operation $\odot$ (i.e. $m^{(2)}$) such that $[xyz]=(x\odot y)\odot z$, i.e.

$$m^{(3)}=m^{(2)}\circ\left(m^{(2)}\times\mathrm{id}\right) \tag{1}$$

for all $x,y,z\in G$, then we say that $[\;]$ (respectively $m^{(3)}$) is *derived* from $\odot$ (respectively from $m^{(2)}$) and denote this fact by $(G,[\;])=\mathrm{der}(G,\odot)$ (respectively by $m^{(3)}=m^{(3)}_{\mathrm{der}}$). If

$$[xyz]=((x\odot y)\odot z)\odot b$$

holds for all $x,y,z\in G$ and some fixed $b\in G$, then a groupoid $(G,[\;]$ is *b-derived* from $(G,\odot)$. In this case we write $(G,[\;])=\mathrm{der}_b(G,\odot)$ (cf. [7, 8]).

We say that $(G,[\;]$ is a *ternary semigroup* if the operation $[\;]$ is *associative*, i.e. if

$$[[xyz]\,uv]=[x\,[yzu]\,v]=[xy\,[zuv]] \tag{2}$$

holds for all $x,y,z,u,v\in G$, i.e.

$$m^{(3)}\circ\left(m^{(3)}\times\mathrm{id}\times\mathrm{id}\right)=m^{(3)}\circ\left(\mathrm{id}\times m^{(3)}\times\mathrm{id}\right)=m^{(3)}\circ\left(\mathrm{id}\times\mathrm{id}\times m^{(3)}\right). \tag{3}$$

Obviously, a ternary operation $m^{(3)}_{\mathrm{der}}$ derived from a binary associative operation $m^{(2)}$ is also associative in the above sense, but a ternary operation $[\;]$ which is b-derived from an associative operation $\odot$ is associative in the above sense, if and only if b lies in the center of $(G,\odot)$.

Fixing one element in a ternary operation we obtain a binary operation. A binary groupoid $(G,\odot)$, where $x\odot y=[xay]$ for some fixed $a\in G$, respectively $\left(G,m^{(2)}_a\right)$, where

$$m^{(2)}_a=m^{(3)}\circ(\mathrm{id}\times a\times\mathrm{id})\,, \tag{4}$$

is called a *retract* of $(G,[\;])$ and is denoted by $\mathrm{ret}_a(G,[\;])$. In some special cases described in [7, 8] we have $(G,\odot)=\mathrm{ret}_a(\mathrm{der}_b(G,\odot))$ and $(G,\odot)=\mathrm{ret}_c(\mathrm{der}_d(G,\odot))$, but in general $(G,\odot)$ and $\mathrm{ret}_a(\mathrm{der}_b(G,\odot))$ are only isomorphic [8].

Lemma 1. *If in the ternary semigroup $(G,[\;])$ there exists an element e such that for all $y\in G$ we have $[eye]=y$, then this semigroup is derived from the binary semigroup $\mathrm{ret}_e(G,[\;])$, i.e. $(G,[\;])=\mathrm{der}(\mathrm{ret}_e(G,[\;]))$, and this semigroup is derived from the binary semigroup $(G,m^{(2)}_e)$, where*

$$m^{(2)}_e=m^{(3)}\circ(\mathrm{id}\times e\times\mathrm{id})\,. \tag{5}$$

Proof. Indeed, if we put $x\circledast y=[xey]$, then $(x\circledast y)\circledast z=[[xey]ez]=[x[eye]z]=[xyz]$ and $x\circledast(y\circledast z)=[xe[yez]]=[x[eye]z]=[xyz]$, which completes the proof. $\square$

The same ternary semigroup $\left(G,m^{(3)}\right)$ can be derived from two different (but isomorphic) semigroups $(G,\circledast)$ and $(G,\diamond)$ $\left(\left(G,m^{(2)}_e\right)\text{ and }\left(G,m^{(2)}_a\right)\right)$. Indeed, if

in G there exists $a \neq e$ such that $[aya] = y$ for all $y \in G$, then by the same argumentation we obtain $[xyz] = x \diamond y \diamond z$ for $x \diamond y = [xay]$. In this case for $\varphi(x) = x \diamond e = [xae]$ we have

$$x \circledast y = [xey] = [x[aea]y] = [[xae]ay] = (x \diamond e) \diamond y = \varphi(x) \diamond y$$

and

$$\varphi(x \circledast y) = [[xey]ae] = [[x[aea]y]ae] = [[xae]a[yae]] = \varphi(x) \diamond \varphi(y).$$

Thus φ is a binary homomorphism such that $\varphi(e) = a$. Moreover, for $\psi(x) = [eax]$ we have

$$\psi(\varphi(x)) = [ea[xae]] = [e[axa]e] = x,$$
$$\varphi(\psi(x)) = [[eax]ae] = [e[axa]e] = x$$

and

$$\psi(x \diamond y) = [ea[xay]] = [ea[x[eae]y]] = [[eax]e[aey]] = \psi(x) \circledast \psi(y).$$

Hence semigroups $(G, \circledast)$ and $(G, \diamond)$ are isomorphic.

Definition 2. An element $e \in G$ is called a *middle identity* or a *middle neutral element* of $(G, [\ ])$, if for all $x \in G$ we have $[exe] = x$, i.e.

$$m^{(3)} \circ (e \times \mathrm{id} \times e) = \mathrm{id}\,. \tag{6}$$

An element $e \in G$ satisfying the identity $[eex] = x$, i.e.

$$m^{(3)} \circ (e \times e \times \mathrm{id}) = \mathrm{id} \tag{7}$$

is called a *left identity* or a *left neutral element* of $(G, [\ ])$. Similarly we define a *right identity*. An element which is a left, middle and right identity is called a *ternary identity* (briefly: identity).

There are ternary semigroups without left (middle, right) neutral elements, but there are also ternary semigroups in which all elements are identities.

Example 1. In ternary semigroups derived from the symmetric group S_3 all elements of order 2 are left and right (but no middle) identities.

Example 2. In ternary semigroup derived from Boolean group all elements are ternary identities, but ternary semigroup 1-derived from the additive group $\mathbb{Z}_4$ has no left (right, middle) identities.

Lemma 3. *For any ternary semigroup $(G, [\ ])$ with a left (right) identity there exists a binary semigroup $(G, \odot)$ and its endomorphism μ such that*

$$[xyz] = x \odot \mu(y) \odot z$$

for all $x, y, z \in G$.

Proof. Let e be a left identity of $(G, [\ ])$. It is not difficult to see that the operation $x \odot y = [xey]$ is associative. Moreover, for $\mu(x) = [exe]$, we have

$$\mu(x) \odot \mu(y) = [[exe]e[eye]] = [[exe][eey]e] = [e[xey]e] = \mu(x \odot y)$$

and

$$[xyz] = [x[eey][eez]] = [[xe[eye]]ez] = x \odot \mu(y) \odot z.$$

In the case of right identity the proof is analogous. □

Definition 4. We say that a ternary groupoid $(G, [\,])$ is:
a *left cancellative* if $[abx] = [aby] \Longrightarrow x = y$,
a *middle cancellative* if $[axb] = [ayb] \Longrightarrow x = y$,
a *right cancellative* if $[xab] = [yab] \Longrightarrow x = y$
holds for all $a, b \in G$.

A ternary groupoid which is left, middle and right cancellative is called *cancellative.*

Theorem 5. *A ternary groupoid is cancellative if and only if it is a middle cancellative, or equivalently, if and only if it is a left and right cancellative.*

Proof. Assume that a ternary semigroup $(G, [\,])$ is a middle cancellative and $[xab] = [yab]$. Then $[ab[xab]] = [ab[yab]]$ and in the consequence $[a[bxa]b] = [a[bya]b]$ which implies $x = y$.

Conversely if $(G, [\ ])$ is a left and right cancellative and $[axb] = [ayb]$ then $[a[axb]b] = [a[ayb]b]$ and $[[aax]bb] = [[aay]bb]$ which gives $x = y$. □

The above theorem is a consequence of the general result proved in [10].

Definition 6. A ternary groupoid $(G, [\,])$ is called σ-commutative, if

$$[x_1 x_2 x_3] = \left[x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}\right] \tag{8}$$

holds for all $x_1, x_2, x_3 \in G$, i.e. if $m^{(3)} = m^{(3)} \circ \sigma$. If (8) holds for all $\sigma \in S_3$, then $(G, [\ ])$ is a *commutative* groupoid. If (8) holds only for $\sigma = (13)$, i.e. if $[x_1 x_2 x_3] = [x_3 x_2 x_1]$, then $(G, [\,])$ is called *semicommutative*.

The group S_3 is generated by two transpositions; (12) and (23). This means that $(G, [\,])$ is commutative if and only if $[xyz] = [yxz] = [xzy]$ holds for all $x, y, z \in G$.

As a simple consequence of Theorem 5 from [9] we obtain

Corollary 7. If in a ternary semigroup $(G, [\,])$ satisfying the identity $[xyz] = [yxz]$ there are a, b such that $[axb] = x$ for all $x \in G$, then $(G, [\,])$ is commutative.

Proof. According to the above remark it is sufficient to prove that $[xyz] = [xzy]$. We have

$$[xyz] = [a[xyz]b] = [ax[yzb]] = [ax[zyb]] = [a[xzy]b] = [xzy].$$

□

Mediality in the binary case is

$$(x \odot y) \odot (z \odot u) = (x \odot z) \odot (y \odot u)\,. \tag{9}$$

This can be presented as a matrix $A^{(2)} = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$, read from left by rows and from top by columns as $\begin{array}{ccc} & \Downarrow & \Downarrow \\ \Rightarrow & x & y \\ \Rightarrow & z & u \end{array}$ (see [1]).

Remark 1. In the binary case a middle cancellative semigroup is commutative, and so for groups mediality coincides with commutativity.

In the ternary case instead of $A^{(2)}$ we have 3×3 matrix $A^{(3)}$ which should be read similarly.

Definition 8. A ternary groupoid $(G,[\,])$ is *medial* if it satisfies the identity

$$[[x_{11}x_{12}x_{13}][x_{21}x_{22}x_{23}][x_{31}x_{32}x_{33}]] = [[x_{11}x_{21}x_{31}][x_{12}x_{22}x_{32}][x_{13}x_{23}x_{33}]],$$

i.e.

$$m^{(3)} \circ \left(m^{(3)} \times m^{(3)} \times m^{(3)}\right) = m^{(3)} \circ \left(m^{(3)} \times m^{(3)} \times m^{(3)}\right) \circ \sigma_{medial}, \tag{10}$$

where $\sigma_{medial} = \binom{123456789}{147258369} \in S_9$.

It is not difficult to see that a semicommutative ternary semigroup is medial.

An element x such that $[xxx] = x$ is called an *idempotent*. A groupoid in which all elements are idempotents is called an *idempotent groupoid*. A left (right, middle) identity is an idempotent.

3. Ternary groups

Definition 9. A ternary semigroup $(G,[\,])$ is a *ternary group* if for all $a,b,c \in G$ there are $x,y,z \in G$ such that

$$[xab] = [ayb] = [abz] = c. \tag{11}$$

One can prove [19] that elements x,y,z are uniquely determined. Moreover, according to the suggestion of [19] one can prove (cf. [6]) that in the above definition, under the assumption of the associativity, it suffices only to postulate the existence of a solution of $[ayb] = c$, or equivalently, of $[xab] = [abz] = c$.

In a ternary group the equation $[xxz] = x$ has a unique solution which is denoted by $z = \overline{x}$ and called *skew element* (cf. [5]), or in the other notation

$$m^{(3)} \circ (\mathrm{id} \times \mathrm{id} \times \bar{\cdot}) \circ D^{(3)} = \mathrm{id},$$

where $D^{(3)}(x) = (x,x,x)$ is a ternary diagonal map and $\bar{\cdot} : x \to \bar{x}$. As a consequence of results obtained in [5] we have

Theorem 10. *In any ternary group $(G,[\,])$ for all $x,y,z \in G$ the following relations take place*

$$\begin{aligned} [x\,x\,\overline{x}] &= [x\,\overline{x}\,x] = [\,\overline{x}\,x\;x] = x, \\ [y\,x\,\overline{x}] &= [y\,\overline{x}\,x] = [x\,\overline{x}\,y] = [\,\overline{x}\,x\,y] = y, \\ \overline{[x\;y\;z]} &= [\,\overline{z}\;\overline{y}\;\overline{x}], \\ \overline{\overline{x}} &= x. \end{aligned}$$

Since in an idempotent ternary group $\overline{x} = x$ for all x, an idempotent ternary group is semicommutative. From the results obtained in [6] (see also [9]) for $n = 3$ we have

Theorem 11. *A ternary semigroup $(G,[\,])$ with a unary operation $^{-} : x \to \overline{x}$ is a ternary group if and only if it satisfies identities*

$$[yx\,\overline{x}\,] = [x\,\overline{x}\,y] = y,$$

or in other notation

$$m^{(3)} \circ (\mathrm{id} \times \mathrm{id} \times \bar{\cdot}) \circ \left(\mathrm{id} \times D^{(2)}\right) = \mathrm{Pr}_1,$$

$$m^{(3)} \circ (\mathrm{id} \times \bar{\cdot} \times \mathrm{id}) \circ \left(D^{(2)} \times \mathrm{id}\right) = \mathrm{Pr}_2,$$

where $D^{(2)}(x) = (x,x)$ and $\mathrm{Pr}_1(x,y) = x$, $\mathrm{Pr}_2(x,y) = y$.

Corollary 12. A ternary semigroup $(G, [\ ])$ is an idempotent ternary group if and only if it satisfies identities

$$[yxx] = [xxy] = y.$$

By Lemma 1 a ternary group with an identity is derived from a binary group.

REMARK 2. The set $S_3 \backslash A_3$ of all odd permutations with ternary operation $[\ ]$ defined as composition of three permutations is an example of a noncommutative ternary group which is not derived from any group (all groups with three elements are commutative and isomorphic to $\mathbb{Z}_3$).

From results proved in [9] follows

Theorem 13. *A ternary group* $(G, [\ ])$ *satisfying the identity*

$$[xy\overline{x}] = y$$

or

$$[\overline{x}yx] = y$$

is commutative.

The most important theorem is

Theorem 14 (Gluskin-Hosszú)**.** *For a ternary group* $(G, [\ ])$ *and fixed element* $a \in G$ *there exist a binary group* $(G, \circledast) = \mathrm{ret}_a\,(G, [\ ])$ *and its automorphism* φ *such that* $\varphi(a) = a$ *and*

$$[xyz] = x \circledast \varphi\,(y) \circledast \varphi^2\,(z) \circledast b, \tag{12}$$

where $b = [\overline{a}\,\overline{a}\,\overline{a}\,]$.

Proof. Let $a \in G$ be fixed. Then the binary operation $x \circledast y = [x\,a\,y]$ is associative, because

$$(x \circledast y) \circledast z = [[xay]az] = [xa[yaz]] = x \circledast (y \circledast z).$$

In $(G, \circledast)$ an element $\overline{a}$ is the identity, $[\overline{a}\,\overline{x}\,\overline{a}]$ inverse of x. $\varphi(x) = [\overline{a}\ x\ a]$ is an automorphism of $(G, \circledast)$. The easy calculation proves that the above formula holds for $b = [\overline{a}\,\overline{a}\,\overline{a}\,]$. (see [21] and [7]). □

One can prove that the group $(G, \circledast)$ is unique up to isomorphism [7]. From the proof of Theorem 3 in [11] it follows that any medial ternary group satisfies the identity

$$\overline{[xyz]} = [\overline{x}\,\overline{y}\,\overline{z}\,],$$

which together with our previous results shows that in such groups we have

$$[\overline{x}\,\overline{y}\,\overline{z}] = [\overline{z}\,\overline{y}\,\overline{x}].$$

But $\overline{\overline{x}} = x$. Hence, any medial ternary group is semicommutative, thus any retract of such group is a commutative group. Moreover, for φ from the proof of Theorem 14 and $\varphi(b) = b$ for $b = [\overline{a}\,\overline{a}\,\overline{a}]$ we have

$$\varphi(\varphi(x)) = [\overline{a}\,[\overline{a}xa]a] = [\overline{a}\,a\,[x\overline{a}\,a]] = x$$

Corollary 15. Any medial ternary group $(G, [\ ])$ has the form

$$[xyz] = x \odot \varphi(y) \odot z \odot b,$$

where $(G, \odot)$ is a commutative group, φ its automorphism such that $\varphi^2 = \mathrm{id}$ and $b \in G$ is fixed.

Corollary 16. A ternary group is medial, if and only if it is semicommutative.

Corollary 17. A ternary group is semicommutative (medial), if and only if there exists $a \in G$ such that $[xay] = [yax]$ holds for all $x, y \in G$.

Corollary 18. A commutative ternary group is b-derived from some commutative group.

Indeed, $\varphi(x) = [\overline{a}\, xa] = [xa\, \overline{a}] = x$.

Theorem 19 (Post)**.** *For any ternary group* $(G, [\,])$ *there exists a binary group* $(G^*, \circledast)$ *and* $H \lhd G^*$*, such that* $G^* \diagup H \simeq \mathbb{Z}_2$ *and*

$$[xyz] = x \circledast y \circledast z$$

for all $x, y, z \in G$*.*

Proof. Let c be a fixed element in G and let $G^* = G \times \mathbb{Z}_2$. In G^* we define binary operation $\circledast$ putting

$$(x, 0) \circledast (y, 0) = ([xy\overline{c}], 1)$$

$$(x, 0) \circledast (y, 1) = ([xyc], 0)$$

$$(x, 1) \circledast (y, 0) = ([xcy], 0)$$

$$(x, 1) \circledast (y, 1) = ([xcy], 1).$$

It is not difficult to see that this operation is associative and $(\overline{c}, 1)$ is its neutral element. The inverse element (in G^*) has the form:

$$(x, 0)^{-1} = (\overline{x}, 0)$$

$$(x, 1)^{-1} = ([\overline{c}\, \overline{x}\, \overline{c}], 1)$$

Thus G^* is a group such that $H = \{(x, 1) : x \in G\} \lhd G^*$. Obviously the set G can be identified with $G \times \{0\}$ and

$$\begin{aligned} x \circledast y \circledast z &= ((x, 0) \circledast (y, 0)) \circledast (z, 0) = ([xy\overline{c}], 1) \circledast (z, 0) \\ &= ([[xy\overline{c}]cz], 0) = ([xy[\overline{c}cz]], 0) = ([xyz], 0) = [xyz], \end{aligned}$$

which completes the proof. □

The original proof of this theorem uses some equivalences of sequences of elements from G (see [19]). Our proof is based on some general method presented in [17]. Note that group G^* satisfying all conditions formulated in our theorem is called *covering* for ternary group $(G, [\,])$. Our construction gives the free covering group in this sense of universal algebras. From results obtained in [8] it follows

Proposition 20. *All retracts of a ternary group* $(G, [\,])$ *are isomorphic to the normal subgroup* H *of* G^* *from the previous theorem, i.e.*

$$\mathrm{ret}_a\,(G, [\,]) \simeq H \lhd G^*.$$

4. Binary representations of ternary groups

For a given ternary group $(G,[\,])$ denote by $(G\times G,*)$ a semigroup with the following binary multiplication

$$(x,y)*(u,v)=([xyu],v)\,. \tag{13}$$

Obviously, for all $x,u,v\in G$ we have $(x,\overline{x})*(u,v)=(\overline{x},x)*(u,v)=(u,v)$, which means that $(x,\overline{x})$ and $(\overline{x},x)$ are left (but not right) unities in $(G\times G,*)$. Generally $(x,\overline{x})\neq(\overline{x},x)$. But for all $x,y\in G$ we have also $(x,y)*(\overline{y},y)=(\overline{y},y)*(x,y)=(x,y)$, i.e. each element (x,y) has a "private" unit. Moreover, any element $(u,\bar{u})$, $u\in G$ is a left unit.

The semigroup $(G\times G,*)$ is left (but not right) cancellative, i.e. $(a,b)*(x,y)=(a,b)*(c,d)$ implies $(x,y)=(c,d)$. Moreover, $(G\times G,*)$ is also a right quasigroup, i.e. for every $(a,b),(c,d)\in G\times G$ there exists only one $(x,y)\in G\times G$ such that $(a,b)*(x,y)=(c,d)$. Similarly it is not difficult to see that for each $a,b,c,d\in G$ there are uniquely determined $x,y\in G$ such that $(x,a)*(b,c)=(a,y)*(b,c)=(d,c)$.

Let V be a vector space over $\mathbb{K}$ and $\operatorname{End}V$ be a set of linear endomorphisms of V.

Definition 21. A *left representation* of a ternary group $(G,[\;])$ in V is a map $\Pi^L:G\times G\to\operatorname{End}V$ such that

$$\Pi^L(x_1,x_2)\circ\Pi^L(x_3,x_4)=\Pi^L([x_1x_2x_3],x_4)\,,\ \ \forall x_1,x_2,x_3,x_4\in G \tag{14}$$

$$\Pi^L(x,\overline{x})=\mathrm{id}_V,\ \ \forall x\in G. \tag{15}$$

Replacing in (15) x by $\overline{x}$ we obtain $\Pi^L(\overline{x},x)=id_V$, which means that in fact (15) has the form $\Pi^L(\overline{x},x)=\Pi^L(x,\overline{x})=\mathrm{id}_V,\ \ \forall x\in G$.

Lemma 22. *For all $x_1,x_2,x_3,x_4\in G$ we have*

$$\Pi^L([x_1x_2x_3],x_4)=\Pi^L(x_1,[x_2x_3x_4])\,.$$

Proof. Indeed, we have

$$\begin{aligned}\Pi^L([x_1x_2x_3]\ ,x_4)&=\Pi^L([x_1x_2x_3]\ ,x_4)\circ\Pi^L(x,\overline{x})\\&=\Pi^L([[x_1x_2x_3]\,x_4\,x]\,,\overline{x})=\Pi^L([x_1\,[x_2x_3x_4]\,x]\,,\overline{x})\\&=\Pi^L(x_1,[x_2x_3x_4])\circ\Pi^L(x,\overline{x})=\Pi^L(x_1,[x_2x_3x_4])\,.\end{aligned}$$

□

Note also that for all $x,y,z\in G$ we have

$$\Pi^L(x,y)=\Pi^L([xz\overline{z}],y)=\Pi^L(x,z)\circ\Pi^L(\overline{z},y) \tag{16}$$

and

$$\Pi^L(x,z)\circ\Pi^L(\overline{z},\overline{x})=\Pi^L(\overline{z},\overline{x})\circ\Pi^L(x,z)=\mathrm{id}_V, \tag{17}$$

i.e. every $\Pi^L(x,z)$ is invertible and $\left(\Pi^L(x,z)\right)^{-1}=\Pi^L(\overline{z},\overline{x})$. This means that any left representation gives a representation of a ternary group by a binary group.

Moreover, if a ternary group $(G,[\,])$ is medial, then

$$\Pi^L(x_1,x_2)\circ\Pi^L(x_3,x_4)=\Pi^L(x_3,x_4)\circ\Pi^L(x_1,x_2)\,,$$

i.e. obtained group is commutative. Indeed, by Corollary 17, we have

$$\begin{aligned}\Pi^L(x_1,x_2)\circ\Pi^L(x_3,x_4) &= \Pi^L(x_1,x_2)\circ\Pi^L(x_3,x_4)\circ\Pi^L(x,\overline{x})\\ &= \Pi^L([[x_1x_2x_3]x_4\,x],\overline{x}) = \Pi^L([[x_3x_4x_1]x_2\,x],\overline{x})\\ &= \Pi^L(x_3,x_4)\circ\Pi^L(x_1,x_2)\circ\Pi^L(x,\overline{x})\\ &= \Pi^L(x_3,x_4)\circ\Pi^L(x_1,x_2)\,.\end{aligned}$$

If $(G,[\,])$ is commutative, then also $\Pi^L(x,y)=\Pi^L(y,x)$, because

$$\begin{aligned}\Pi^L(x,y) &= \Pi^L(x,y)\circ\Pi^L(x,\overline{x}) = \Pi^L([x\,y\,x],\overline{x})\\ &= \Pi^L([y\,x\,x],\overline{x}) = \Pi^L(y,x)\circ\Pi^L(x,\overline{x}) = \Pi^L(y,x)\,.\end{aligned}$$

Thus in the case of commutative and idempotent ternary groups any left representation is idempotent and, in the consequence, $\left(\Pi^L(x,y)\right)^{-1}=\Pi^L(x,y)$. This means that commutative and idempotent ternary groups are represented by boolean groups.

Proposition 23. *Let* $(G,[\,])=\operatorname{der}(G,\odot)$ *be a ternary group derived from a binary group* $(G,\odot)$*. There is one-to-one correspondence between representations of* $(G,\odot)$ *and left representations of* $(G,[\,])$*.*

Proof. Because $(G,[\,])=\operatorname{der}(G,\odot)$, then $x\odot y=[xey]$ and $\overline{e}=e$, where e is unity of the binary group $(G,\odot)$. If $\pi\in\operatorname{Rep}(G,\odot)$, then (as it is not difficult to see) $\Pi^L(x,y)=\pi(x)\circ\pi(y)$ is a left representation of $(G,[\,])$. Conversely, if Π^L is a left representation of $(G,[\,])$ then $\pi(x)=\Pi^L(x,e)$ is a representation of $(G,\odot)$. Moreover, in this case $\Pi^L(x,y)=\pi(x)\circ\pi(y)$. Indeed, by Lemma 22, we have

$$\Pi^L(x,y)=\Pi^L(x,[eye])=\Pi^L([xey],e)=\Pi^L(x,e)\circ\Pi^L(y,e)=\pi(x)\circ\pi(y)$$

for all $x,y\in G$. □

Let $(G,[\,])$ be a ternary group and $(G\times G,*)$ be a semigroup used to the construction of left representations. According to Post [19] we say that two pairs (a,b), (c,d) of elements of G are equivalent, if there exists an element $x\in G$ such that $[abx]=[cdx]$. Using a covering group we can see that if this equation holds for some $x\in G$, then it holds also for all $x\in G$. This means that

$$\Pi^L(a,b)=\Pi^L(c,d)\Longleftrightarrow(a,b)\sim(c,d),$$

i.e.

$$\Pi^L(a,b)=\Pi^L(c,d)\Longleftrightarrow[abx]=[cdx]$$

for some $x\in G$. Indeed, if $[abx]=[cdx]$ holds for some $x\in G$, then

$$\begin{aligned}\Pi^L(a,b) &= \Pi^L(a,b)\circ\Pi^L(x,\overline{x})=\Pi^L([abx],\overline{x})\\ &= \Pi^L([cdx],\overline{x})=\Pi^L(c,d)\circ\Pi^L(x,\overline{x})=\Pi^L(c,d).\end{aligned}$$

The converse is obvious.

Now we consider the second construction. Let $(G,[\,])$ be a ternary group. On $G\times G$ we define the following binary operation

$$(x,y)\diamond(u,v)=(u,[vxy])$$

Then $(G \times G, \diamond)$ is a binary semigroup which is isomorphic to $(G \times G, *)$. This isomorphism has the form $\varphi((x,y)) = (\overline{y}, \overline{x})$. Indeed,

$$\begin{aligned}\varphi((x,y) \diamond (u,v)) &= \varphi(\,(u,[vxy])\,) = (\,\overline{[vxy]}, \overline{u}\,) \\ &= (\,[\,\overline{y}, \overline{x}, \overline{v}\,], \overline{u}\,) = (\,\overline{y}, \overline{x}\,) * (\,\overline{v}, \overline{u}\,) = \varphi((x,y)) * \varphi((u,v)).\end{aligned}$$

Basing on this construction we can define

Definition 24. A *right representation* of a ternary group G in V is a map $\Pi^R : G \times G \to \operatorname{End} V$ such that

$$\Pi^R(x_3,x_4) \circ \Pi^R(x_1,x_2) = \Pi^R(x_1,[x_2x_3x_4]), \quad \forall x_1,x_2,x_3,x_4 \in G \tag{18}$$

$$\Pi^R(x,\overline{x}) = \mathrm{id}_V, \quad \forall x \in G. \tag{19}$$

From (18)-(19) it follows that

$$\Pi^R(x,y) = \Pi^R(x,[z\,\overline{z}\,y]) = \Pi^R(\overline{z},y) \circ \Pi^R(x,z) \quad \forall\ x,y,z \in G. \tag{20}$$

It is easy to check that $\Pi^R(x,y) = \Pi^L(\overline{y},\overline{x}) = \left(\Pi^L(x,y)\right)^{-1}$. So it is enough to consider only left representations (as in binary case).

Example 3. Let G be a ternary group and $\mathbb{K}G$ is a vector space spanned by G, which means that any element of $\mathbb{K}G$ can be uniquely presented in the form $u = \sum_{i=1}^n k_i y_i$, $k_i \in \mathbb{K}$, $y_i \in G$. Then left and right regular representations are defined by

$$\Pi^L_{reg}(x_1,x_2)\,u = \sum_{i=1}^n k_i\,[x_1x_2y_i]\,, \tag{21}$$

$$\Pi^R_{reg}(x_1,x_2)\,u = \sum_{i=1}^n k_i\,[y_ix_1x_2]\,, \tag{22}$$

5. Middle representations

Now we build another type of representations using the following construction. For a given ternary group $(G,[\ ])$ we define on $G \times G^{op}$, where G^{op} is a ternary group having opposite multiplication, the following ternary operation $\langle\ \rangle$ putting

$$\langle (x_1,y_1),(x_2,y_2),(x_3,y_3)\rangle = (\,[x_1\,x_2\,x_3],\,[y_3\,y_2\,y_1]\,) \tag{23}$$

for all $x_1,x_2,x_3,y_1,y_2,y_3 \in G$. It is not difficult to see that $(G \times G, \langle\ \rangle)$ is a ternary group as a direct product of ternary groups. This group is commutative (medial, idempotent), if and only if $(G,[\ ])$ is commutative (respectively: medial, idempotent). It is clear that

$$\langle (x,y),(\overline{x},\overline{y}),(a,b)\rangle = \langle (a,b),(x,y),(\overline{x},\overline{y})\rangle = (a,b)$$

$$\langle (\overline{x},\overline{y}),(x,y),(a,b)\rangle = \langle (a,b),(\overline{x},\overline{y}),(x,y)\rangle = (a,b)$$

for all $x,y,a,b \in G$. This means that in the group $(G \times G, \langle\ \rangle)$ the element skew to (x,y) has the form $(\overline{x},\overline{y})$, where $\overline{x}$ is skew in $(G,[\ ])$.

Using (23) we construct the middle representations as follows.

Definition 25. A *middle representation* of a ternary group G in V is a map $\Pi^{M}:G\times G\rightarrow \operatorname{End}V$ such that

$$\Pi^{M}\left(x_3,y_3\right)\circ\Pi^{M}\left(x_2,y_2\right)\circ\Pi^{M}\left(x_1,y_1\right)=\Pi^{M}\left(\left[x_3x_2x_1\right],\left[y_1y_2y_3\right]\right),\quad\forall x_1,x_2,x_3,y_1,y_2,y_3\in G, \tag{24}$$

$$\Pi^{M}\left(x,y\right)\circ\Pi^{M}\left(\overline{x},\overline{y}\right)=\Pi^{M}\left(\overline{x},\overline{y}\right)\circ\Pi^{M}\left(x,y\right)=\mathrm{id}_V\quad\forall\, x,y\in G. \tag{25}$$

It is seen that a middle representation is a ternary group homomorphism $\Pi^{M}: G\times G^{op}\rightarrow \operatorname{der}\operatorname{End}V$. Note that instead of (25) one can use $\Pi^{M}\left(x,\overline{y}\right)\circ\Pi^{M}\left(\overline{x},y\right)=id_V$ after changing x to $\overline{x}$ and taking into account that $x=\overline{\overline{x}}$.

REMARK 3. In case elements x and y are idempotent we have $\Pi^{M}(x,y)\circ\Pi^{M}(x,y)=id_V$, which means that the matrices Π^{M} are Boolean. Thus all middle representation matrices of idempotent ternary groups are Boolean.

In general, the composition $\Pi^{M}\left(x_1,y_1\right)\circ\Pi^{M}\left(x_2,y_2\right)$ is not a middle representation, but the following proposition holds.

Proposition 26. *If Π^{M} is a middle representation of a ternary group $(G,[\ ])$, then for any fixed $z\in G$*

1. Let $\Pi^{L}_{z}(x,y)=\Pi^{M}(x,z)\circ\Pi^{M}(y,\overline{z})$ is a left representation of $(G,[\ ])$, then $\Pi^{L}_{z}(x,y)\circ\Pi^{L}_{z'}(x',y')=\Pi^{L}_{z'}([xyz'],y')$.

2. Let $\Pi^{R}_{z}(x,y)=\Pi^{M}(z,y)\circ\Pi^{M}(\overline{z},x)$ is a right representation of $(G,[\ ])$, then $\Pi^{R}_{z}(x,y)\circ\Pi^{R}_{z'}(x',y')=\Pi^{R}_{z}(x,[yx'y'])$.

Proof. The proof is a verification of the corresponding axioms. □

In particular, Π^{L}_{z} (Π^{R}_{z}) is a family of left (right) representations.

Corollary 27. If a middle representation Π^{M} of a ternary group $(G,[\ ])$ satisfies $\Pi^{M}\left(x,\overline{x}\right)=\mathrm{id}_V$ for all $x\in G$, then it is a left and right representation and $\Pi^{M}(x,y)=\Pi^{M}(y,x)$ for all $x,y\in G$.

Proof. Indeed,

$$\begin{aligned}\Pi^{M}(x,y)&=\Pi^{M}([xy\overline{y}],[y\overline{z}z])\\&=\Pi^{M}(x,z)\circ\Pi^{M}(y,\overline{z})\circ\Pi^{M}(\overline{y},y)=\Pi^{M}(x,z)\circ\Pi^{M}(y,\overline{z})=L(x,y).\end{aligned}$$

Similarly

$$\begin{aligned}\Pi^{M}(x,y)&=\Pi^{M}([z\overline{z}x],[\overline{x}xy])\\&=\Pi^{M}(z,y)\circ\Pi^{M}(\overline{z},x)\circ\Pi^{M}(x,\overline{x})=\Pi^{M}(z,y)\circ\Pi^{M}(\overline{z},x)=R(x,y)\end{aligned}$$

and

$$\begin{aligned}\Pi^{M}(x,y)&=\Pi^{M}(\,[\,x\,y\,\overline{y}\,],[\,y\,x\,\overline{x}\,]\,)\\&=\Pi^{M}(x,\overline{x})\circ\Pi^{M}(y,x)\circ\Pi^{M}(\overline{y},y)=\Pi^{M}(y,x),\end{aligned}$$

which completes the proof. □

Observe that in general $\Pi^{M}_{reg}(x,\overline{x})\neq\mathrm{id}$.

It can be shown that for regular representations we have the following commutation relations

$$\Pi^{L}_{reg}\left(x_1,y_1\right)\circ\Pi^{R}_{reg}\left(x_2,y_2\right)=\Pi^{R}_{reg}\left(x_2,y_2\right)\circ\Pi^{L}_{reg}\left(x_1,y_1\right).$$

Proposition 28. *For a finite (or countable) ternary group $(G, [\,])$ left and right representations are unitary.*

Proof. Take a scalar product $\langle\ ,\ \rangle$ in $\mathbb{K}G$ which makes G an orthonormal basis, i.e. $\langle g, h\rangle = \delta_{g,h}$. Then the unitarity follows from uniqueness of solutions to the group equations $[xyg] = h$ (see (11)). □

6. Relation between representations

Let $(G, [\,])$ be a ternary group and let $(G \times G, \langle\,\rangle)$ be a ternary group used to the construction of the middle representation. In $(G, [\,])$ (and in the consequence in $(G \times G, \langle\,\rangle)$) we define the relation

$$(a, b) \sim (c, d) \Longleftrightarrow [azb] = [czd]$$

for all $z \in G$. It is not difficult to see that this relation is a congruence in $(G\times G, \langle\,\rangle)$. For regular representations $\Pi^M_{reg}(a, b) = \Pi^M_{reg}(c, d)$ if $(a, b) \sim (c, d)$.

Thus in Example 4 we have $\Pi^M(a, b) = \Pi^M(c, d) \Longleftrightarrow (a + b) = (c + d) \bmod 3$. Hence, the computation of middle representations can be reduced to the computation only of three cases $\Pi^M(0, 0)$, $\Pi^M(1, 0)$, $\Pi^M(2, 0)$.

So we have the following relation

$$a \precsim a' \Longleftrightarrow a = [\overline{x} a' x] \text{ for some } x \in G$$

or equivalently

$$a \precsim a' \Longleftrightarrow a' = [x a \overline{x}] \text{ for some } x \in G.$$

It is not difficult to see that it is an equivalence relation on $(G, [\,])$, moreover, if $(G, [\,])$ is medial, then this relation is a congruence.

Let $(G \times G, \langle\,\rangle)$ be a ternary group used for a construction of middle representations, then

$$\begin{aligned}(a, b) \precsim (a', b) \Longleftrightarrow a' = [x a \overline{x}] &\text{ and} \\ b' = [y b \overline{b}\,] &\text{ for some } (x, y) \in G \times G\end{aligned}$$

is an equivalence relation on $(G \times G, \langle\,\rangle)$. Moreover, if $(G, [\,])$ is medial, then this relation is a congruence. Unfortunately, it is a weak relation. In a ternary group $\mathbb{Z}_3$, where $[xyz] = (x - y + z) \,(\mathrm{mod}\, 3)$ we have only one class, i.e. all elements are equivalent. In $\mathbb{Z}_4$ with the operation $[xyz] = (x + y + z + 1) \,(\mathrm{mod}\ 4)$ we have $a \precsim a' \Longleftrightarrow a = a'$. But for this relation holds the following

Lemma 29. *If $(a, b) \precsim (a', b')$, then*

$$\operatorname{tr} \Pi^M(a, b) = \operatorname{tr} \Pi^M(a', b').$$

Proof. We have $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $A, B \in \mathrm{End}V$. Indeed,

$$\begin{aligned}\operatorname{tr} \Pi^M(a, b) &= \operatorname{tr} \Pi^M([x a' \overline{x}], [y b' \overline{y}]\,) = \operatorname{tr}\left(\Pi^M(x, \overline{y}) \circ \Pi^M(a', b') \circ \Pi^M(\overline{x}, y)\right) \\ &= \operatorname{tr}\left(\Pi^M(x, \overline{y}) \circ \Pi^M(\overline{x}, y) \circ \Pi^M(a', b')\right) = \operatorname{tr}\left(id_V \circ \Pi^M(a'b')\right) \\ &= \operatorname{tr} \Pi^M(a', b')\end{aligned}$$

□

We can "algebralize" the above regular representations from the Example 4 in the following way. From (14) we have for the left representation $\Pi_{reg}^{L}(i,j)\circ\Pi_{reg}^{L}(k,l)=\Pi_{reg}^{L}(i,[jkl])$, where $[jkl]=j-k+l$, $i,j,k,l\in\mathbb{Z}_3$. Denote $\gamma_i^L=\Pi_{reg}^{L}(0,i)$, $i\in\mathbb{Z}_3$, then we obtain the algebra with the relations $\gamma_i^L\gamma_j^L=\gamma_{i+j}^L$. Conversely, any matrix representation of $\gamma_i\gamma_j=\gamma_{i+j}$ leads to the left representation by $\Pi^L(i,j)=\gamma_{j-i}$.

In the case of the middle regular representation we introduce $\gamma_{k+l}^M=\Pi_{reg}^{M}(k,l)$, $k,l\in\mathbb{Z}_3$, then we obtain

$$\gamma_i^M\gamma_j^M\gamma_k^M=\gamma_{[ijk]}^M,\quad i,j,k\in\mathbb{Z}_3. \tag{26}$$

In some sense (26) can be treated as a *ternary analog of Clifford algebra*. As before, any matrix representation of (26) gives the middle representation $\Pi^M(k,l)=\gamma_{k+l}$.

In our *derived* case the connection with the standard group representations is given by

Proposition 30. *Let $(G,\odot)$ be a binary group, and the ternary derived group as $(G,[\,])=\mathrm{der}\,(G,\odot)$. There is one-to-one correspondence between a pair of commuting binary groups representations and a middle ternary derived group representation.*

Proof. Let $\pi,\rho\in Rep\,(G,\odot)$, $\pi(x)\circ\rho(y)=\rho(y)\circ\pi(x)$ and $\Pi^L\in Rep\,(G,[\,])$. We take

$$\Pi^M(x,y)=\pi(x)\circ\rho\left(y^{-1}\right),$$
$$\pi(x)=\Pi^M(x,e),$$
$$\rho(x)=\Pi^M(e,\overline{x}).$$

Then using (24) we prove the needed representation laws. □

Let $(G,[\,])$ be a fixed ternary group, $(G\times G,\langle\,\rangle)$ a corresponding ternary group used in the construction of middle representations, $((G\times G)^*,\circledast)$ a covering group of $(G\times G,\langle\,\rangle)$, $(G\times G,\diamond)=ret_{(a,b)}(G\times G,\langle\,\rangle)$. If $\Pi^M(a,b)$ is a middle representation of $(G,[\,])$, then π defined by

$$\pi(x,y,0)=\Pi^M(x,y)$$

and

$$\pi(x,y,1)=\Pi^M(x,y)\circ\Pi^M(a,b)$$

is a representation of the covering group. Moreover

$$\rho(x,y)=\Pi^M(x,y)\circ\Pi^M(a,b)=\pi(x,y,1)$$

is a representation of the above retract induced by (a,b). Indeed, $(\overline{a},\overline{b})$ is the identity of this retract and $\rho(\overline{a},\overline{b})=\Pi^M(\overline{a},\overline{b})\circ\Pi^M(a,b)=\mathrm{id}_V$. Similarly

$$\begin{aligned}\rho\left((x,y)\diamond(z,u)\right)&=\rho\left(\langle(x,y),(a,b),(z,u)\rangle\right)=\rho\left([xaz],[uby]\right)\\&=\Pi^M\left([xaz],[uby]\right)\circ\Pi^M(a,b)\\&=\Pi^M(x,y)\circ\Pi^M(a,b)\circ\Pi^M(z,u)\circ\Pi^M(a,b)\\&=\rho(x,y)\circ\rho(z,u)\end{aligned}$$

But $\tau(x)=(x,\overline{x})$ is an embedding of $(G,[\ ])$ into $(G\times G,\langle\ \rangle)$. Hence μ defined by $\mu(x,0)=\Pi^M(x,\overline{x})$ and $\mu(x,1)=\Pi^M(x,\overline{x})\circ\Pi^M(a,\overline{a})$ is a representation of a covering group G^* for $(G,[\ ])$ (see Post theorem for $a=c$). On the

other hand, $\beta(x) = \Pi^M(x,\overline{x}) \circ \Pi^M(a,\overline{a})$ is a representation of a binary retract $(G,\cdot\,) = ret_a(G,[\;])$. That β can induce some middle representation of $(G,[\;])$ (by Gluskin-Hosszú theorem).

Note that in a ternary group of quaternions $(\mathbb{K},[\;])$, where $[xyz] = xyz(-1) = -xyz$ and xy is a multiplication of quaternions (-1 is a central element) we have $\overline{1} = -1$, $\overline{-1} = 1$ and $\overline{x} = x$ for others. In $(K \times K, \langle\;\rangle)$ we have $(a,b) \sim (-a,-b)$ and $(a,-b) \sim (-a,b)$, which gives 32 two-elements equivalence classes. The embedding $\tau(x) = (x,\overline{x})$ suggest that $\Pi^M(i,i) = \pi(i) \neq \pi(-i) = \Pi^M(-i,-i)$. Generally $\Pi^M(a,b) \neq \Pi^M(-a,-b)$ and $\Pi^M(a,-b) \neq \Pi^M(-a,b)$.

The relation $(a,b) \sim (c,d) \iff [abx] = [cdx]$ for all $x \in G$ is a congruence on $(G \times G, *)$. Note that this relation can be defined as "for some x". Indeed, using a covering group we can see that if $[abx] = [cdx]$ holds for some x then holds also for all x. Thus $\pi^L(a,b) = \Pi^L(c,d) \iff (a,b) \sim (c,d)$. Indeed

$$\begin{aligned}\Pi^L(a,b) &= \Pi^L(a,b) \circ \Pi^L(x,\overline{x}) = \Pi^L([a\ b\ x],\overline{x})) \\ &= \Pi^L([c\ d\ x],\overline{x})) = \Pi^L(c,d) \circ \Pi^L(x,\overline{x}) = \Pi^L(c,d).\end{aligned}$$

Proposition 31. *Every left representation of a commutative group* $(G,[\;])$ *is a middle representation.*

Proof. Indeed,

$$\begin{aligned}\Pi^L(x,y) \circ \Pi^L(\overline{x},\overline{y}) &= \Pi^L([x\ y\ \overline{x}],\overline{y}) \\ &= \Pi^L([x\ \overline{x}\ y],\overline{y}) = \Pi^L(y,\overline{y}) = \mathrm{id}_V\end{aligned}$$

and

$$\begin{aligned}\Pi^L(x_1,x_2) \circ \Pi^L(x_3,x_4) \circ \Pi^L(x_5,x_6) &= \Pi^L([[x_1x_2x_3]x_4x_5],x_6) \\ &= \Pi^L([[x_1x_3x_2]x_4x_5],x_6) = \Pi^L([x_1x_3[x_2x_4x_5]],x_6)\end{aligned}$$

$$= \Pi^L([x_1x_3[x_5x_4x_2]],x_6) = \Pi^L([x_1x_3x_5],[x_4x_2x_6]) = \Pi^L([x_1x_3x_5],[x_6x_4x_2]).$$

□

Note that the converse holds only for middle representations such that $\Pi^M(x,\overline{x}) = \mathrm{id}_V$.

Theorem 32. *There is one-one-correspondence between left representations of* $(G,[\;])$ *and binary representations of the retract* $ret_a(G,[\;])$.

Proof. Let $\Pi^L(x,a)$ is given, then define $\rho(x) = \Pi^L(x,a)$ is such representation of the retract which can be directly shown. Conversely, assume that $\rho(x)$ is a representation of the retract $ret_a(G,[\;])$. Define $\Pi^L(x,y) = \rho(x) \circ \rho(\overline{y})^{-1}$, then $\Pi^L(x,y) \circ \Pi^L(z,u) = \rho(x) \circ \rho(\overline{y})^{-1} \circ \rho(z) \circ \rho(\overline{u})^{-1} = \rho(x \circledast (\overline{y})^{-1} \circ \circledast z) \circ \rho(\overline{u})^{-1} = \rho([\,[\,x\,a\,[\,\overline{a}\,y\,\overline{a}\,]\,]\,a\,z\,]) \circ \rho(\overline{u})^{-1} = \rho([\,x\,y\,x\,]) \circ \rho(\overline{u})^{-1} = \Pi^L([\,x\,y\,z\,],u)$ which completes the proof. □

Remark 4. It is seen that Proposition 23 is a direct consequence of this theorem.

7. Matrix representations

Now we give examples of matrix representations for concrete ternary groups.

EXAMPLE 4. Let $G = \mathbb{Z}_3 \ni \{0,1,2\}$ and the ternary multiplication is $[xyz] = x - y + z$. Then $[xyz] = [zyx]$ and $\overline{0} = 0$, $\overline{1} = 1$, $\overline{2} = 2$, therefore $(G, [\ ])$ is an idempotent medial ternary group. Thus $\Pi^L(x,y) = \Pi^R(y,x)$ and

$$\Pi^L(a,b) = \Pi^L(c,d) \Longleftrightarrow (a-b) = (c-d) \operatorname{mod} 3. \tag{27}$$

Straightforward calculations give the left regular representation in the manifest matrix form

$$\begin{aligned}
\Pi^L_{reg}(0,0) &= \Pi^L_{reg}(2,2) = \Pi^L_{reg}(1,1) = \Pi^R_{reg}(0,0) \\
&= \Pi^R_{reg}(2,2) = \Pi^R_{reg}(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= [1] \oplus [1] \oplus [1], \\
\Pi^L_{reg}(2,0) &= \Pi^L_{reg}(1,2) = \Pi^L_{reg}(0,1) = \Pi^R_{reg}(2,1) \\
&= \Pi^R_{reg}(1,0) = \Pi^R_{reg}(0,2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\
&= [1] \oplus \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = [1] \oplus \left[-\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right] \oplus \left[-\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right], \\
\Pi^L_{reg}(2,1) &= \Pi^L_{reg}(1,0) = \Pi^L_{reg}(0,2) = \Pi^R_{reg}(2,0) \\
&= \Pi^R_{reg}(1,2) = \Pi^R_{reg}(0,1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
&= [1] \oplus \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = [1] \oplus \left[-\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right] \oplus \left[-\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right].
\end{aligned}$$

Consider the middle representation constructions of Examples 3 and 4.

EXAMPLE 5. The middle regular representations from Example 3 is defined by

$$\Pi^M_{reg}(x_1, x_2)\, u = \sum_{i=1}^{n} k_i\, [x_1 y_i x_2]$$

For regular representations we have

$$\Pi^M_{reg}(x_1, y_1) \circ \Pi^R_{reg}(x_2, y_2) = \Pi^R_{reg}(y_2, y_1) \circ \Pi^M_{reg}(x_1, x_2)\,, \tag{28}$$

$$\Pi^M_{reg}(x_1, y_1) \circ \Pi^L_{reg}(x_2, y_2) = \Pi^L_{reg}(x_1, x_2) \circ \Pi^M_{reg}(y_2, y_1)\,. \tag{29}$$

Example 6. For the middle regular representation matrices we obtain

$$\Pi_{reg}^{M}\left(0,0\right)=\Pi_{reg}^{M}\left(1,2\right)=\Pi_{reg}^{M}\left(2,1\right)=\left(\begin{array}{ccc}1&0&0\\0&0&1\\0&1&0\end{array}\right),$$

$$\Pi_{reg}^{M}\left(0,1\right)=\Pi_{reg}^{M}\left(1,0\right)=\Pi_{reg}^{M}\left(2,2\right)=\left(\begin{array}{ccc}0&1&0\\1&0&0\\0&0&1\end{array}\right),$$

$$\Pi_{reg}^{M}\left(0,2\right)=\Pi_{reg}^{M}\left(2,0\right)=\Pi_{reg}^{M}\left(1,1\right)=\left(\begin{array}{ccc}0&0&1\\0&1&0\\1&0&0\end{array}\right).$$

Example 7. The above representation Π_{reg}^{M} of $(\mathbb{Z}_3,[\,])$ is equivalent to the orthogonal direct sum of two irreducible representations

$$\Pi_{reg}^{M}\left(0,0\right)=\Pi_{reg}^{M}\left(1,2\right)=\Pi_{reg}^{M}\left(2,1\right)=[1]\oplus\left[\begin{array}{cc}-1&0\\0&1\end{array}\right],$$

$$\Pi_{reg}^{M}\left(0,1\right)=\Pi_{reg}^{M}\left(1,0\right)=\Pi_{reg}^{M}\left(2,2\right)=[1]\oplus\left[\begin{array}{cc}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{array}\right],$$

$$\Pi_{reg}^{M}\left(0,2\right)=\Pi_{reg}^{M}\left(2,0\right)=\Pi_{reg}^{M}\left(1,1\right)=[1]\oplus\left[\begin{array}{cc}\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{array}\right],$$

i.e. one-dimensional trivial [1] and two-dimensional irreducible.

Remark 5. In this example $\Pi^{M}(x,\overline{x})=\Pi^{M}(x,x)\neq id_V$, but $\Pi^{M}(x,y)\circ\Pi^{M}(x,y)=id_V$, and so Π^{M} are of second degree.

Let us consider a more complicated example of left representations.

Example 8. Let $G=\mathbb{Z}_4\ni\{0,1,2,3\}$ and the ternary multiplication is

$$[xyz]=(x+y+z+1)\bmod 4. \tag{30}$$

We have the multiplication table

$$[x,y,0]=\left(\begin{array}{cccc}1&2&3&0\\2&3&0&1\\3&0&1&2\\0&1&2&3\end{array}\right)\qquad[x,y,1]=\left(\begin{array}{cccc}2&3&0&1\\3&0&1&2\\0&1&2&3\\1&2&3&0\end{array}\right)$$

$$[x,y,2]=\left(\begin{array}{cccc}3&0&1&2\\0&1&2&3\\1&2&3&0\\2&3&0&1\end{array}\right)\qquad[x,y,3]=\left(\begin{array}{cccc}0&1&2&3\\1&2&3&0\\2&3&0&1\\3&0&1&2\end{array}\right)$$

Then the skew elements are $\overline{0}=3$, $\overline{1}=2$, $\overline{2}=1$, $\overline{3}=0$, therefore $(G,[\,])$ is an (nonidempotent) commutative ternary group. The left representation is defined by expansion $\Pi_{reg}^{L}\left(x_1,x_2\right)u=\sum_{i=1}^{n}k_i\left[x_1x_2y_i\right]$, which means that

$$\Pi_{reg}^{L}\left(x,y\right)|z>=|\,[xyz]>.$$

Analogously, for right and middle representations

$$\Pi^{R}_{reg}(x,y)\,|z>=|\,[zxy]>,\quad \Pi^{M}_{reg}(x,y)\,|z>=|\,[xzy]>.$$

Therefore $|\,[xyz]>=|\,[zxy]>=|\,[xzy]>$ and

$$\Pi^{L}_{reg}(x,y)=\Pi^{R}_{reg}(x,y)\,|z>=\Pi^{M}_{reg}(x,y)\,|z>,$$

so $\Pi^{L}_{reg}(x,y)=\Pi^{R}_{reg}(x,y)=\Pi^{M}_{reg}(x,y)$. Thus it is sufficient to consider the left representation only.

In this case the equivalence is $\Pi^{L}(a,b)=\Pi^{L}(c,d)\iff(a+b)=(c+d)\bmod 4$, and we obtain the following classes

$$\Pi^{L}_{reg}(0,0)=\Pi^{L}_{reg}(1,3)=\Pi^{L}_{reg}(2,2)=\Pi^{L}_{reg}(3,1)$$
$$=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}=[1]\oplus[-1]\oplus[-i]\oplus[i],$$

$$\Pi^{L}_{reg}(0,1)=\Pi^{L}_{reg}(1,0)=\Pi^{L}_{reg}(2,3)=\Pi^{L}_{reg}(3,2)$$
$$=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}=[1]\oplus[-1]\oplus[-1]\oplus[-1],$$

$$\Pi^{L}_{reg}(0,2)=\Pi^{L}_{reg}(1,1)=\Pi^{L}_{reg}(2,0)=\Pi^{L}_{reg}(3,3)$$
$$=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}=[1]\oplus[-1]\oplus[i]\oplus[-i],$$

$$\Pi^{L}_{reg}(0,3)=\Pi^{L}_{reg}(1,2)=\Pi^{L}_{reg}(2,1)=\Pi^{L}_{reg}(3,0)$$
$$=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=[1]\oplus[-1]\oplus[1]\oplus[1].$$

It is seen that, due to the fact that the ternary operation (30) is commutative, there are only one-dimensional irreducible left representations.

In a similar way one can extend other notions of the classical group representation theory to the ternary group case. This includes, e.g. direct sum and tensor product of representations, characters, irreducibility (Schur lemma), equivalence of representations etc.

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